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Iterative approximation of fixed points of nonexpansive mappings

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Abstract

Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and $T : K \to K$ be a nonexpansive mapping with $F(T) := \{x \in K: Tx = x\} \neq \emptyset$. For a fixed $\delta \in (0, 1)$, define $S : K \to K$ by $Sx := (1 - \delta)x + \delta Tx$, $\forall x \in K$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of K which satisfies $z_t = tu + (1 - t)Tz_t$ for arbitrary $u \in K$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ which satisfies the following conditions: $C1$: $\lim \alpha_n = 0$; $C2$: $\sum \alpha_n = \infty$. For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n.$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Keywords: Uniformly Gâteaux differentiable norm; Uniformly smooth real Banach spaces

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1. Introduction

Let E be a real Banach space, K a closed convex subset of E and $T : K \to K$ a nonexpansive mapping. For fixed $t \in (0, 1)$ and arbitrary $u \in K$, let $z_t \in K$ denote the unique fixed point of T_t defined by $T_t x := tu + (1 - t)Tx$, $x \in K$. Assume $F(T) := \{x \in K: Tx = x\} \neq \emptyset$. In 1967, Browder [2] proved that if $E = H$, a Hilbert space, then $\lim_{t\to 0} z_t$ exists and is a fixed point of T. In 1980, Reich [7] extended this result to uniformly smooth Banach spaces. In 1981, Kirk [4] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range. The following theorem is now well known.

Theorem. (See, e.g., [6,7]) *Let K be a nonempty closed convex subset of a Banach space E which has uniformly Gâteaux differentiable norm and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \to z_t$, $0 < t < 1$ satisfying $z_t = tu + (1 - t)Tz_t$, for arbitrary but fixed $u \in K$, which converges to a fixed point of T.*

For a sequence $\{\alpha_n\}$ of real numbers in $[0, 1]$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n. \tag{1}$$

The recursion formula (1) was first introduced in 1967 by Halpern [3] in the framework of Hilbert spaces. He proved the weak convergence of $\{x_n\}$ to a fixed point of T where $\alpha_n := n^{-a}$, $a \in (0, 1)$. In 1977, Lions [5] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1: \quad \lim \alpha_n = 0; \qquad C2: \quad \sum \alpha_n = \infty; \qquad C3: \quad \lim \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$$

In 1980, Reich [7] proved that the result of Halpern remains true when E is uniformly smooth.

It was observed that both Halpern's and Lion's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n + 1)$. This was overcome in 1992 by Wittmann [11] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$C1: \quad \lim \alpha_n = 0; \qquad C2: \quad \sum \alpha_n = \infty; \qquad C3: \quad \sum |\alpha_{n+1} - \alpha_n| < \infty.$$

Reich [8] extended the result of Wittmann to Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps (e.g., l_p spaces, $1 < p < \infty$), where $\{\alpha_n\}$ satisfies $C1$ and $C2$ of Wittmann and is also required to be decreasing (and hence also satisfies $C3$). These spaces exclude L_p spaces, $1 < p < \infty$, $p \neq 2$. Recently, Shioji and Takahashi [9] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset of K has the fixed point property for nonexpansive mappings (e.g., L_p spaces, $1 < p < \infty$). In particular, they proved the following theorem.

Theorem *(***ST** [9]*). Let E be a real Banach space whose norm is uniformly Gâteaux differentiable and let K be a closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T)$: $\{x \in K\colon\ Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence which satisfies the following conditions*:

(i) $0 \leqslant \alpha_n \leqslant 1$, $\lim \alpha_n = 0$;
(ii) $\sum \alpha_n = \infty$;
(iii) $!\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Let $u \in K$ and let $\{x_n\}$ be defined by $x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geqslant 0.$$

Assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \to 0$, where for $0 < t < 1$, z_t is the unique element of K which satisfies $z_t = tu + (1 - t)Tz_t$. Then, $\{x_n\}$ converges strongly to z.

In 2002, Xu [13] (see also [12]) improved the result of Lion twofold. First, he weakened the condition $C3$ by removing the square in the denominator so that the canonical choice of $\alpha_n = 1/(n + 1)$ is possible. Secondly, he proved the strong convergence of the scheme (1) in the framework of real uniformly smooth Banach spaces. In particular, he proved the following theorem.

Theorem *(***HKX1** [13, Theorem 3.1]*). Let E be a uniformly smooth real Banach space, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping with a fixed point. Let $u, x_0 \in K$ be given. Assume that $\{\alpha_n\} \subset [0, 1]$ satisfies the conditions*:

(1) $\lim \alpha_n = 0$;
(2) $\sum \alpha_n = \infty$;
(3) $\lim \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0$.

Then the sequence $\{x_n\}$ generated by $x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geqslant 0,$$

converges strongly to a fixed point of T.

He showed that condition $C3$ of Lion and his condition (3) are not comparable. Xu also remarked [13, Remark 3.2] that Halpern [3] observed that conditions (1) and (2) are necessary for the strong convergence of algorithm (1) for all nonexpansive mappings, $T : K \to K$. It is unclear if they are sufficient. This brings us to the following question.

Question. Are the conditions (i) $\lim \alpha_n = 0$ and, (ii) $\sum \alpha_n = \infty$ sufficient for the strong convergence of algorithm (1) for all nonexpansive mappings $T : K \to K$?

It is our purpose in this paper to prove a significant improvement of Theorems HKX1 and ST in the following sense. First, we prove the strong convergence of the algorithm (1) in the framework of real Banach spaces E with uniformly Gâteaux differentiable norms and without condition (iii) of Theorem ST. Our theorem then also extends Theorem HKX1 to the more general real Banach spaces with uniformly Gâteaux differentiable norms and at the same time dispenses with condition (iii) of that theorem. Consequently, we give an affirmative answer to the above question.

2. Preliminaries

Let $S := \{x \in E\colon \|x\| = 1\}$ denote the unit sphere of the real Banach space E. E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; and E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$, the limit is attained uniformly for $x \in S$. Let E be a normed space with $\dim E \geqslant 2$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1\colon\ \|x\| = 1;\ \|y\| = \tau\right\}.$$

The space E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \rho_E(t)/t = 0$.

We shall make use of the following well-known result.

Lemma 2.1. *Let E be a real normed linear space. Then, the following inequality holds*:

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E,\ \forall j(x + y) \in J(x + y).$$

In the sequel, we shall also make use of the following lemmas.

Lemma 2.2. (Suzuki [10]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in* $[0,1]$ *with* $0 < \liminf \beta_n \leqslant \limsup \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ *for all integers* $n \geqslant 0$ *and* $\limsup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. *Then,* $\lim \|y_n - x_n\| = 0$.

Lemma 2.3. (Xu [13]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leqslant (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geqslant 0,$$

where,

(i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \leqslant 0$;
(iii) $\gamma_n \geqslant 0$; $(n \geqslant 0)$, $\sum \gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

In the sequel, $F(T) := \{x \in K\colon Tx = x\}$. We prove the following theorem.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For a fixed $\delta \in (0,1)$, define $S : K \to K$ by $Sx := (1-\delta)x + \delta Tx$, $\forall x \in K$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of K which satisfies $z_t = tu + (1-t)Tz_t$ for arbitrary $u \in K$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ which satisfies the following conditions: $C1$: $\lim \alpha_n = 0$; $C2$: $\sum \alpha_n = \infty$. For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n. \tag{2}$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Observe first that S is nonexpansive and has the same set of fixed points as T. Define

$$\beta_n := (1-\delta)\alpha_n + \delta, \quad \forall n \geqslant 0; \qquad y_n := \frac{x_{n+1} - x_n + \beta_n x_n}{\beta_n}, \quad n \geqslant 0. \tag{3}$$

Observe also that $\beta_n \to \delta$ as $n \to \infty$, and that if $\{x_n\}$ is bounded, then $\{y_n\}$ is bounded. Let $x^* \in F(T) = F(S)$. One easily shows by induction that $\|x_n - x^*\| \leqslant \max\{\|x_0 - x^*\|, \|u - x^*\|\}$ for all integers $n \geqslant 0$, and so, $\{x_n\}$, $\{y_n\}$, $\{Tx_n\}$ and $\{Sx_n\}$ are all bounded. Also,

$$\|x_{n+1} - Sx_n\| = \alpha_n \|u - Sx_n\| \to 0, \tag{4}$$

as $n \to \infty$. Observe also that from the definitions of β_n and S, we obtain that $y_n = (\alpha_n u + (1-\alpha_n)\delta Tx_n)/\beta_n$ so that

$$\begin{aligned}\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leqslant{}& \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| . \|u\| + \frac{(1-\alpha_{n+1})}{\beta_{n+1}}\delta\|Tx_{n+1} - Tx_n\| \\ &+ \left|\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\right|\delta\|Tx_n\| - \|x_{n+1} - x_n\|,\end{aligned}$$

so that, since $\{x_n\}$ and $\{Tx_n\}$ are bounded, we obtain that, for some constants $M_1 > 0$, and $M_2 > 0$,

$$\begin{aligned}&\limsup\big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \\ &\quad\leqslant \limsup\left\{\left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| . \|u\| + \left|\frac{(1-\alpha_{n+1})}{\beta_{n+1}}\delta - 1\right| M_1 \right. \\ &\qquad \left. + \left|\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\right|\delta M_2\right\} \leqslant 0.\end{aligned}$$

Hence, by Lemma 2.2, $\|y_n - x_n\| \to 0$ as $n \to \infty$. Consequently, $\lim \|x_{n+1} - x_n\| = \lim \beta_n \|y_n - x_n\| = 0$. Combining this with (4) yields that

$$\|x_n - Sx_n\| \to 0 \quad \text{as } n \to \infty. \tag{5}$$

We now show that

$$\limsup\langle u-z, j(x_n-z)\rangle \leqslant 0. \tag{6}$$

For each integer $n \geqslant 0$, let $t_n \in (0,1)$ be such that

$$t_n \to 0 \quad \text{and} \quad \frac{\|x_n - Sx_n\|}{t_n} \to 0, \quad n \to \infty. \tag{7}$$

Let $z_{t_n} \in K$ be the unique fixed point of the contraction mapping S_{t_n} given by

$$S_{t_n}x = t_n u + (1-t_n)Sx, \quad x \in K.$$

Then,

$$z_{t_n} - x_n = t_n(u - x_n) + (1-t_n)(Sz_{t_n} - x_n).$$

Using the inequality of Lemma 2.1, we compute as follows:

$$\begin{aligned}
\|z_{t_n} - x_n\|^2 &\leqslant (1-t_n)^2\|Sz_{t_n} - x_n\|^2 + 2t_n\langle u - x_n, j(z_{t_n} - x_n)\rangle \\
&\leqslant (1-t_n)^2\big(\|Sz_{t_n} - Sx_n\| + \|Sx_n - x_n\|\big)^2 + 2t_n\big(\|z_{t_n} - x_n\|\big)^2 \\
&\quad + \langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle \\
&\leqslant \big(1+t_n^2\big)\|z_{t_n} - x_n\|^2 + \|Sx_n - x_n\|\big(2\|z_{t_n} - x_n\| + \|Sx_n - x_n\|\big) \\
&\quad + 2t_n\langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \\
&\quad \leqslant \frac{t_n}{2}\|z_{t_n} - x_n\|^2 + \frac{\|Sx_n - x_n\|}{2t_n} \times \big(2\|z_{t_n} - x_n\| + \|Sx_n - x_n\|\big).
\end{aligned}$$

Since $\{x_n\}$, $\{z_{t_n}\}$ and $\{Sx_n\}$ are bounded and $\|Sx_n - x_n\|/2t_n \to 0$, $n \to \infty$, it follows from the last inequality that

$$\limsup\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leqslant 0. \tag{8}$$

Moreover, we have that

$$\begin{aligned}
\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle &= \langle u - z, j(x_n - z)\rangle + \langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \\
&\quad + \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle.
\end{aligned} \tag{9}$$

But, by hypothesis, $z_{t_n} \to z \in F(S)$, $n \to \infty$. Thus, using the boundedness of $\{x_n\}$, we obtain that

$$\langle z - z_{t_n}, j(x_n - z_{t_n})\rangle \to 0, \quad n \to \infty.$$

Also,

$$\langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \to 0, \quad n \to \infty,$$

since j is norm-to-weak* uniformly continuous on bounded subsets of E. Hence, we obtain from (8) and (9) that

$$\limsup\langle u - z, j(x_n - z\rangle \leqslant 0.$$

Furthermore, from (2) we get that $x_{n+1} - z = \alpha_n(u - z) + (1 - \alpha_n)(Sx_n - z)$. It then follows that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leqslant (1 - \alpha_n)^2\|Sx_n - z\|^2 + 2\alpha_n\langle u - z, j(x_{n+1} - z)\rangle \\ &\leqslant (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n\sigma_n, \end{aligned}$$

where $\sigma_n := 2\langle u - z, j(x_{n+1} - z)\rangle$; $\gamma_n \equiv 0$, $\forall n \geqslant 0$. Thus, by Lemma 2.3, $\{x_n\}$ converges strongly to a fixed point of T. □

Remark 3.2. We note that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm and is such that every nonempty closed convex and bounded subset of E has the fixed point property for nonexpansive maps (see, e.g., [1]).

Remark 3.3. Theorem 3.1 is a significant generalization of Theorems ST and HKX1 as has been explained in the introduction. Furthermore, our method of proof which is different from the method of Shioji and Takahashi [9] is of independent interest.

Let $S_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} S^k x$. With this definition, Xu also proved the following theorem.

Theorem *(***HKX2** [13, Theorem 3.2]*). Assume that E is a real uniformly convex and uniformly smooth Banach space. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)S_n x_n, \quad n \geqslant 0. \tag{10}$$

Assume that

(i) $\lim \alpha_n = 0$;
(ii) $\sum \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to a fixed point of $S : K \to K$ nonexpansive.

Remark 3.4. Theorem 3.1 is also a significant improvement of Theorem HKX2 in the sense that the recursion formula (2) is simpler and requires less computer time than the recursion formula (10). Moreover, the requirement that E be also uniformly convex imposed in Theorem HKX2 is dispensed with in Theorem 3.1. Furthermore, Theorem 3.1 is proved in the framework of the more general real Banach spaces with uniformly Gâteaux differentiable norms.

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